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The C''_{pk} index for asymmetric tolerances: Implications and inference

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Abstract. The process capability index C_{pk} has been widely used in the manufacturing industry to provide numerical measures on process performance. Since C_{pk} is a yield-based index which is independent of the target T, it fails to account for process centering with symmetric tolerances, and presents an even greater problem with asymmetric tolerances. Pearn and Chen (1998) considered a new generalization C''_{pk} which was shown to be superior to other existing generalizations of C_{pk} for processes with asymmetric tolerances. In this paper, we investigate the relation between the fraction nonconforming and the value of C''_{pk}. Furthermore, we derive explicit forms of the cumulative distribution function and the probability density function for the natural estimator $\hat{C}''_{pk}$, under the assumption of normality. We also develop a decision making rule based on the natural estimator $\hat{C}''_{pk}$, which can be used to test whether the process is capable or not.

Key words: Asymmetric tolerance; Decision making rule; Normally distributed process; Process yield.

1 Introduction

Process capability indices (PCIs), providing numerical measures of whether or not the ability of a manufacturing process meets a preset level of production tolerance, have recently been a research focus in quality assurance literature. Examples include Kane (1986), Chan *et al.* (1988), Zhang *et al.* (1991), Boyles (1991 and 1994), Pearn *et al.* (1992), Vännman and Kotz (1995), Vännman(1997), Pearn and Chen (1998), Chen *et al.* (1999), and Kotz and Johnson (2002).

Among various capability indices that have been introduced, C_{pk} is defined as

$$C_{pk} = \frac{\min\{USL - \mu,\ \mu - LSL\}}{3\sigma}, \tag{1}$$

which can be alternatively written as:

$$C_{pk} = \frac{d - |\mu - m|}{3\sigma}, \tag{2}$$

where μ is the process mean, σ is the process standard deviation, USL and LSL are the upper and the lower specification limits, respectively, $m = (USL + LSL)/2$, and $d = (USL - LSL)/2$. The index C_{pk} has been widely used in the manufacturing industry, and provides a measure of process yield. In fact, we can calculate the process yield as

$$2\Phi(3C_{pk}) - 1 < \%\text{Yield} < \Phi(3C_{pk})$$

if the process is normally distributed, where $\Phi(\cdot)$ is the cumulative function for the standard normal distribution. To investigate the relationship between the capability indices and the process yield, Boyles (1994) considered the index S_{pk}, a generalization of C_{pk}, which is defined as:

$$S_{pk} = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{USL - \mu}{\sigma}\right) + \frac{1}{2}\Phi\left(\frac{\mu - LSL}{\sigma}\right)\right\}, \tag{3}$$

where Φ^{-1} is the inverse function of Φ. For normally distributed process, the index S_{pk} is a one-to-one transformation of fraction nonconforming (percentage of defective items). We note that given $S_{pk} = c$, we can calculate the process yield as $2\Phi(3c) - 1$. Therefore, S_{pk} represents the actual process yield unlike C_{pk} which is only approximately related to process yield (Boyles (1994)).

A process is said to have a symmetric tolerance if the target value T is the midpoint of the specification interval (LSL, USL), i.e. $T = m$. Although cases with symmetric tolerances are common in practical situations, cases with asymmetric tolerances ($T \neq m$) often occur in the manufacturing industry.

Since both C_{pk} and S_{pk} are independent of target value T, then they do not take into account the asymmetry of the tolerance. Both C_{pk} and S_{pk} fail to distinguish between on-target and off-target processes for processes with asymmetric tolerances. Hence, both C_{pk} and S_{pk} cannot provide consistent and reasonable measures on process capability for processes with asymmetric tolerances. To overcome the problem, Pearn and Chen (1998) proposed the index C''_{pk} which was shown to be superior to other existing generalizations of C_{pk} for processes with asymmetric tolerances. Under the assumption of normality, Pearn and Chen (1998) considered the natural estimator $\hat{C}''_{pk}$ of C''_{pk}, and obtained the exact formula for the r-th moment. In this paper, we investigate the relation between the fraction nonconforming and the value of C''_{pk}. Furthermore, we derive explicit forms of the cumulative distribution function and the probability density function of the natural estimator $\hat{C}''_{pk}$, under the assumption of normality. We show that the cumulative distribution function and the probability density function of the natural estimator $\hat{C}''_{pk}$ can be expressed in terms of a mixture of the chi-square distribution and the normal distribution. The explicit forms of the cumulative distribution function and the probability density function considerably simplify the complexity for analyzing the statistical properties of the natural estimator $\hat{C}''_{pk}$. We also analyze the bias and the MSE of the natural estimator $\hat{C}''_{pk}$ for normally distributed processes with an asymmetric tolerance. In addition, we develop a decision making rule based on the natural estimator $\hat{C}''_{pk}$, which can be used to test whether the process is capable or not.

2 The generalization C''_{pk}

Pearn and Chen (1998) proposed index C''_{pk}, a generalization of C_{pk} for processes with asymmetric tolerances. The generalization C''_{pk} is defined as:

$$C''_{pk} = \frac{d^* - A^*}{3\sigma}, \tag{4}$$

where $D_u = USL - T$, $D_\ell = T - LSL$, $d^* = \min\{D_u, D_\ell\}$, $A^* = \max\{d^*(\mu - T)/D_u,\ d^*(T - \mu)/D_\ell\}$. Obviously, if $T = m$ (symmetric tolerance), then $d^* = D_u = D_\ell = d$, $A^* = |\mu - m|$ and C''_{pk} reduces to the original index C_{pk}. We note that the index C''_{pk} obtains the maximal values at $\mu = T$, regardless of whether the preset specification tolerances are symmetric or asymmetric.

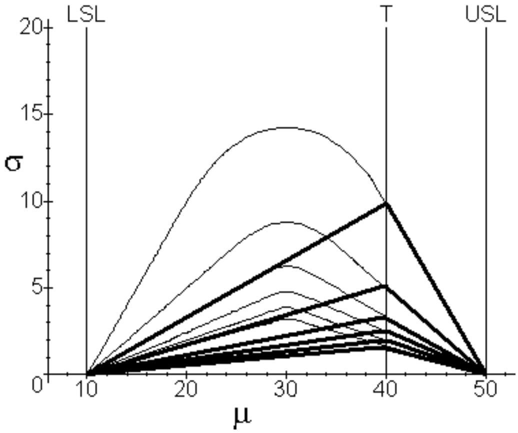

Fig. 1. Contours of $C''_{pk}(\mu, \sigma) = c$ (bold) and $S_{pk}(\mu, \sigma) = S_{pk}(T, d^*/(3c))$ (thin) for $c = 1/3, 2/3, 1, 4/3, 5/3$, and 2 (top to bottom in plot) with (LSL, T, USL) = (10, 40, 50).

If $\mu = T$, then $A^* = 0$ and $C''_{pk} = d^*/(3\sigma)$. Therefore, if $\mu = T$ and $C''_{pk} = c$, then $\sigma = d^*/(3c)$. Since both C''_{pk} and S_{pk} are functions of (μ, σ), we denote them by $C''_{pk}(\mu, \sigma)$ and $S_{pk}(\mu, \sigma)$. Figures 1 and 2 display the contours of $C''_{pk}(\mu, \sigma) = c$ (bold) and $S_{pk}(\mu, \sigma) = S_{pk}(T, d^*/(3c))$ (thin) for $c = 1/3, 2/3, 1, 4/3, 5/3$, and 2 (top to bottom in plots) with asymmetric cases (LSL,T, USL) = (10, 40, 50) and (LSL, T, USL) = (10,34, 50), i.e., $D_\ell : d : D_u = 3 : 2 : 1$ and $D_\ell : d : D_u = 6 : 5 : 4$, respectively. Since $C''_{pk} \leq S_{pk}(T, d^*/(3C''_{pk}))$ for all values of (μ, σ), we conclude that given a process with $C''_{pk}(\mu, \sigma) = c$ the fraction nonconforming would be guaranteed to be less than that of a process with $S_{pk}(\mu, \sigma) = S_{pk}(T, d^*/(3c))$ which is $2\{1 - \Phi[3S_{pk}(T, d^*/(3c))]\}$. For a given threshold value of C''_{pk}, we note that these contours are used to form boundaries, separating the acceptable values from the unacceptable values of (μ, σ). In addition, we have

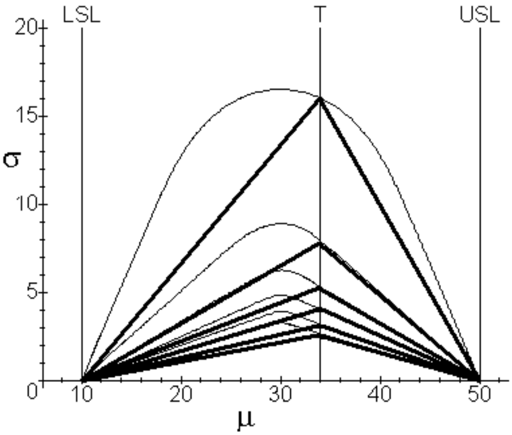

Fig. 2. Contours of $C''_{pk}(\mu, \sigma) = c$ (bold) and $S_{pk}(\mu, \sigma) = S_{pk}(T, d^*/(3c))$ (thin) for c = 1/3, 2/3, 1, 4/3, 5/3, and 2 (top to bottom in plot) with (LSL, T, USL) = (10, 34, 50).

$$S_{pk}\left(T, \frac{d^*}{3c}\right) = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{3c}{\min\{1,r\}}\right) + \frac{1}{2}\Phi(3c\max\{1,r\})\right\},$$

where $r = D_\ell/D_u$. For example, c = 1 with r = 3 gives the fraction nonconforming would be guaranteed to be less than $2\{1 - \Phi[3S_{pk}(T, d^*/(3c))]\} = 2 - [\Phi(3c\ /\min\{1,r\}) + \Phi(3c\max\{1,r\})] = 2 - [\Phi(3) + \Phi(9)] = 1350\,\mathrm{PPM}$ for the asymmetric case (LSL, T, USL) = (10, 40, 50) in Figure 1, c = 1 with $r = 3/2$ gives the fraction nonconforming would be guaranteed to be less than $2 - [\Phi(3) + \Phi(9/2)] = 1353\,\mathrm{PPM}$ for the asymmetric case (LSL, T, USL) = (10, 34, 50) in Figure 2, where "PPM" denotes the "Parts-Per-Million".

3 Distribution of $\hat{C}''_{pk}$

To estimate the generalization C''_{pk}, Pearn and Chen (1998) considered the natural estimator $\hat{C}''_{pk}$ defined in the following. The natural estimator $\hat{C}''_{pk}$ is obtained by replacing the process mean μ and the process variance σ^2 by their conventional estimators $\overline{X}$ and S^2, which may be obtained from a stable process.

$$\hat{C}''_{pk} = \frac{d^* - \hat{A}^*}{3S}, \tag{5}$$

where $d^* = \min\{D_u, D_\ell\}$, $\hat{A}^* = \max\{d^*(\overline{X} - T)/D_u, \quad d^*(T - \overline{X})/D_\ell\}$, $\overline{X} = \sum_{i=1}^{n} X_i/n$ and $S = \{(n-1)^{-1}\sum_{i=1}^{n}(X_i - \overline{X})^2\}^{1/2}$. Obviously, if T = m (symmetric tolerance), then $d^* = D_u = D_\ell = d$, $\hat{A}^* = |\overline{X} - m|$ and $\hat{C}''_{pk}$ reduces to $\hat{C}_{pk}$, the natural estimator of C_{pk} discussed by Kotz *et al* (1993).

We now define $B = n^{1/2}(d^*/\sigma)$, $K = (n-1)S^2/\sigma^2$, $Z = n^{1/2}(\overline{X} - T)/\sigma$, $Y = [\max\{(d^*/D_u)Z, -(d^*/D_\ell)Z\}]^2$, $\delta = n^{1/2}(\mu - T)/\sigma$. Then, the estimator $\hat{C}''_{pk}$ can be written as:

$$\hat{C}''_{pk} = \frac{\sqrt{n-1}(B - \sqrt{Y})}{3\sqrt{nK}}. \tag{6}$$

Under the assumption of normality of X, K is distributed as χ^2_{n-1}, a chi-square distribution with n − 1 degrees of freedom, and Z is distributed as the normal distribution $N(\delta, 1)$ with mean δ and variance 1. Let $\Phi(\cdot)$ and $\phi(\cdot)$ be the cumulative distribution function and the probability density function of the standard normal distribution $N(0, 1)$, respectively. Then, the cumulative distribution function and the probability density function of Z can be expressed as: $\Phi(z - \delta)$ and $\phi(z - \delta)$, respectively. Hence, the cumulative distribution function of Y can be expressed as:

$$F_Y(y) = \Phi[(D_u/d^*)\sqrt{y} - \delta] - \Phi[-(D_\ell/d^*)\sqrt{y} - \delta]. \tag{7}$$

The probability density function of Y can be expressed as:

$$f_Y(y) = \frac{1}{2d^*\sqrt{y}}\left(D_u\phi[(D_u/d^*)\sqrt{y} - \delta] + D_\ell\phi[(D_\ell/d^*)\sqrt{y} + \delta]\right). \tag{8}$$

$F_Y(y)$ and $f_Y(y)$ can be used to derive the sampling distribution of $\hat{C}''_{pk}$ (see Appendix A).

Therefore, the cumulative distribution function of $\hat{C}''_{pk}$ can be obtained as the following.

$$F_{\hat{C}''_{pk}}(x) = \begin{cases} \int_{B^2}^{\infty} F_K(L(x, y)) f_Y(y)dy, & x < 0, \\ 1 - F_Y(B^2), & x = 0, \\ 1 - \int_0^{B^2} F_K(L(x, y)) f_Y(y)dy, & x > 0, \end{cases} \tag{9}$$

and the probability density function of $\hat{C}''_{pk}$ can be derived as:

$$f_{\hat{C}''_{pk}}(x) = \begin{cases} \int_1^{\infty} f_K\left(L(x, B^2t)\right) f_Y\left(B^2t\right) \frac{2L(x, B^2t)}{-x} dt, & x < 0, \\ \int_0^1 f_K\left(L(x, B^2t)\right) f_Y\left(B^2t\right) \frac{2L(x, B^2t)}{x} dt, & x > 0, \end{cases} \tag{10}$$

where $B = n^{1/2}(d^*/\sigma)$, $L(x, y) = (n - 1)(B - y^{1/2})^2/(9nx^2)$, $F_K(\cdot)$ is the cumulative distribution function of K, $f_K(\cdot)$ is the probability density function of K, $F_Y(\cdot)$ is the cumulative distribution function of Y expressed as Eq. (7), and $f_Y(\cdot)$ is the probability density function of Y expressed as Eq. (8).

As an illustration for some of the results obtained, we plot the probability density functions of $\hat{C}''_{pk}$ for an asymmetric case ($D_\ell : d : D_u = 6 : 5 : 4$) with $\sigma = d^*/3$, $\xi = -1.0, -0.5, 0.5, 1.0$, and $n = 10, 20, 50$, where $\xi = (\mu - T)/\sigma$ and $d^* = \min\{D_u, D_\ell\}$. Figures 3 and 4 display the plots of the probability density functions of $\hat{C}''_{pk}$ for $\xi = -1.0 (C''_{pk} = 0.78)$ and $\xi = 1.0 (C''_{pk} = 0.67)$, respectively. From Figures 3 and 4 we observe that for n = 10 the distributions are skew and have large spread. We also observe that as n increases the spread decreases and so does the skewness. We also observe that the estimated index $\hat{C}''_{pk}$ is approximately unbiased for sample size $n > 50$.

Pearn and Chen (1998) derived the *r*-th moment of $\hat{C}''_{pk}$ without using the distribution of $\hat{C}''_{pk}$. We note that the estimator $\hat{C}''_{pk}$ is biased. The magnitude of the bias is $B(\hat{C}''_{pk}) = E(\hat{C}''_{pk}) - C''_{pk}$. The mean square error can be expressed as $MSE(\hat{C}''_{pk}) = Var(\hat{C}''_{pk}) + B^2(\hat{C}''_{pk})$, where $Var(\hat{C}''_{pk}) = E(\hat{C}''_{pk})^2 - E^2(\hat{C}''_{pk})$ is

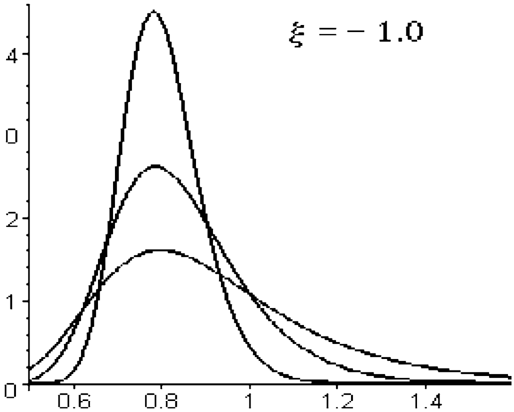

Fig. 3. The pdf of $\hat{C}''_{pk}$ with $\sigma = d^*/3$, $\xi = -1.0$, and $n = 10$ (bottom), 20 (middle), and 50 (top) for the asymmetric case $D_\ell : d : D_u = 6 : 5 : 4$.

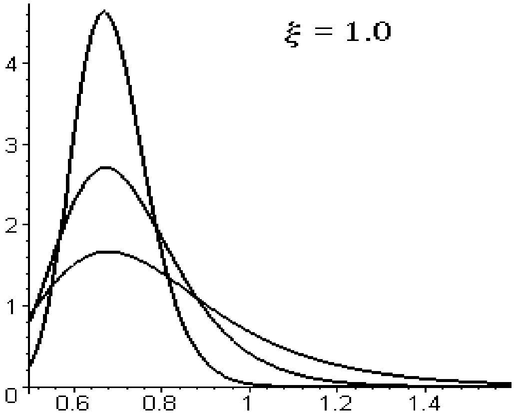

Fig. 4. The pdf of $\hat{C}''_{pk}$ with $\sigma = d^*/3$, $\xi = 1.0$, and $n = 10$ (bottom), 20 (middle), and 50 (top) for the asymmetric case $D_\ell : d : D_u = 6 : 5 : 4$.

the variance of $\hat{C}''_{pk}$. To investigate the behavior of the estimator $\hat{C}''_{pk}$, the bias and the mean square error are calculated (using *Maple* computer software) for various values of $\xi = (\mu - T)/\sigma$, $b = d^*/\sigma$, $d_\ell = d/D_\ell$, $d_u = d/D_u$, and sample size n.

Tables 1, 2, and 3 display the values of C''_{pk}, $B(\hat{C}''_{pk})$ and $MSE(\hat{C}''_{pk})$ for $\xi = -1.0\ (0.5)1.0$, $(d_\ell, d_u) = (5/6, 5/4)$, and $n = 10(10)50$, with $b = 3$, 4, and 5, respectively. We note that the specification with $(d_\ell, d_u) = (5/6, 5/4)$ is asymmetric.

From Tables 1, 2, and 3, we observe that as the sample size n increases, both the bias and the mean square error of $\hat{C}''_{pk}$ decrease. Figure 5 displays the plot of the bias of $\hat{C}''_{pk}$ (vs. n) with $\xi = 0$, 1.0, and -1.0 (from bottom to top in the plot) for fixed $b = 3$, $d_\ell = 5/6$, $d_u = 5/4$. Figure 6 displays the plot of the MSE of $\hat{C}''_{pk}$ (vs. n) with $\xi = 1.0, -1.0$, and 0 (from bottom to top in the plot) for fixed $b = 3$, $d_\ell = 5/6$, $d_u = 5/4$.

Table 1. The values of C''_{pk} , $B(\hat{C}''_{pk})$ and $MSE(\hat{C}''_{pk})$ for $b = 3$, $\xi = -1.0(0.5)1.0$, $d_\ell = 5/6$, $d_u = 5/4$, and $n = 10(10)50$

n	$\xi = -1.0$		$\xi = -0.5$		$\xi = 0$		$\xi = 0.5$		$\xi = 1.0$	
	bias	MSE	bias	MSE	bias	MSE	bias	MSE	bias	MSE
10	0.0733	0.0651	0.0791	0.0806	0.0175	0.0807	0.0739	0.0785	0.0628	0.0575
20	0.0325	0.0234	0.0366	0.0295	−0.0099	0.0311	0.0342	0.0296	0.0278	0.0214
30	0.0209	0.0141	0.0237	0.0179	−0.0147	0.0195	0.0223	0.0181	0.0179	0.0131
40	0.0154	0.0101	0.0175	0.0128	−0.0160	0.0142	0.0164	0.0130	0.0132	0.0094
50	0.0122	0.0079	0.0139	0.0099	−0.0162	0.0113	0.0130	0.0101	0.0104	0.0073
C''_{pk}	0.7778		0.8889		1.0000		0.8333		0.6667	

Table 2. The values of C''_{pk}, $B(\hat{C}''_{pk})$ and $MSE(\hat{C}''_{pk})$ for $b = 4$, $\xi = -1.0(0.5)1.0$, $d_\ell = 5/6$, $d_u = 5/4$, and $n = 10(10)50$

n	$\xi = -1.0$		$\xi = -0.5$		$\xi = 0$		$\xi = 0.5$		$\xi = 1.0$	
	bias	MSE	bias	MSE	bias	MSE	bias	MSE	bias	MSE
10	0.1047	0.1264	0.1105	0.1485	0.0490	0.1474	0.1053	0.1427	0.0942	0.1115
20	0.0464	0.0449	0.0505	0.0535	0.0041	0.0551	0.0482	0.0523	0.0418	0.0403
30	0.0298	0.0270	0.0327	0.0322	−0.0058	0.0341	0.0312	0.0317	0.0268	0.0244
40	0.0220	0.0192	0.0241	0.0230	−0.0094	0.0248	0.0230	0.0227	0.0198	0.0175
50	0.0174	0.0150	0.0191	0.0179	−0.0110	0.0195	0.0182	0.0177	0.0156	0.0136
C''_{pk}	1.1111		1.2222		1.3333		1.1667		1.0000	

Table 3. The values of C''_{pk}, $B(\hat{C}''_{pk})$ and $MSE(\hat{C}''_{pk})$ for $b = 5$, $\xi = -1.0(0.5)1.0$, $d_\ell = 5/6$, $d_u = 5/4$, and $n = 10(10)50$

n	$\xi = -1.0$		$\xi = -0.5$		$\xi = 0$		$\xi = 0.5$		$\xi = 1.0$	
	bias	MSE	bias	MSE	bias	MSE	bias	MSE	bias	MSE
10	0.1361	0.2092	0.1419	0.2380	0.0804	0.2357	0.1367	0.2286	0.1256	0.1871
20	0.0603	0.0739	0.0644	0.0851	0.0180	0.0867	0.0621	0.0826	0.0557	0.0669
30	0.0387	0.0444	0.0416	0.0511	0.0032	0.0532	0.0401	0.0499	0.0358	0.0403
40	0.0285	0.0316	0.0307	0.0365	−0.0028	0.0385	0.0296	0.0356	0.0263	0.0288
50	0.0226	0.0246	0.0243	0.0283	−0.0058	0.0302	0.0235	0.0277	0.0209	0.0224
C''_{pk}	1.4444		1.5556		1.6667		1.5000		1.3333	

From Tables 1, 2, and 3, we also observe that as the value of b increases, both the bias and the mean square error of $\hat{C}''_{pk}$ increase for fixed d_ℓ, d_u, ξ, and n. Figure 7 displays the plot of the bias of $\hat{C}''_{pk}$ (vs. n) with b = 3, 4, and 5 (from bottom to top in the plot) for fixed $\xi = 0.5$, $d_\ell = 5/6$, $d_u = 5/4$. Figure 8 displays the plot of the MSE of $\hat{C}''_{pk}$ (vs. n) with b = 3, 4, and 5 (from bottom to top in the plot) for fixed $\xi = 0.5$, $d_\ell = 5/6$, $d_u = 5/4$.

We note that $\hat{C}''_{pk}$ is a biased estimator. The results in Tables 1–3, Figures 5 and 7 indicate that the bias of $\hat{C}''_{pk}$ is positive when $\mu \neq T$. That is, C''_{pk} is generally overestimated by $\hat{C}''_{pk}$. On the other hand, when $\mu = T$, we have $A^* = 0$ and $C''_{pk} = d^*/(3\sigma)$, the bias of $\hat{C}''_{pk}$ tends to be negative for some cases as shown in Tables 1–3 and Figure 5. Thus, $\hat{C}''_{pk}$ is smaller than C''_{pk} and the bias is negative when $\mu = T$. This is partially contributed by the fact that $\hat{A}^*$

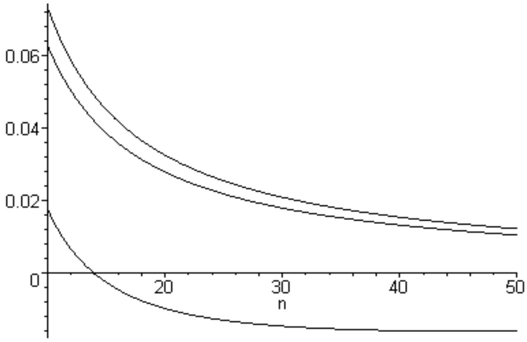

Fig. 5. Bias plot of $\hat{C}''_{pk}$ (vs. n) for b = 3, d_ℓ = 5/6, d_u = 5/4 with ξ = 0, 1.0, and –1.0 (from bottom to top in the plot).

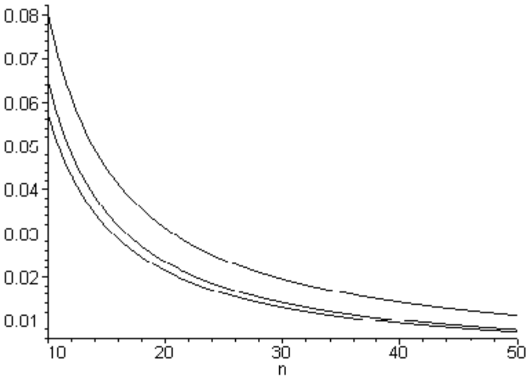

Fig. 6. MSE plot of C''_{pk} (vs. n) for b = 3, d_ℓ = 5/6, d_u = 5/4 with ξ = 1.0, -1.0, and 0 (from bottom to top in the plot).

is calculated to be positive (see Eq.(5)) even when $\mu =$ T and $A^* = 0$. Clearly, the presence of $\hat{A}^*$ in Eq. (5) reduces the value of the calculated $\hat{C}''_{pk}$. As the sample size n increases, the mean square error of $\hat{C}''_{pk}$ decreases. Proper sample sizes for capability estimation are essential. The smaller the sample size is, the higher the value of $\hat{C}''_{pk}$ is required to justify the true process capability.

4 A decision making rule for testing C''_{pk}

Using the index C''_{pk}, the engineers can access the process performance and monitor the manufacturing processes on routine basis. To obtain a decision making rule we consider a testing hypothesis with the null hypothesis $C''_{pk} \le C$ (the process is incapable) and the alternative hypothesis $C''_{pk} > C$ (the process is capable). The null hypothesis will be rejected if $\hat{C}''_{pk} > c_\alpha$, where the constant c_α, called the critical value, is determined so that the significance level of the test is α, i.e., $P(\hat{C}''_{pk} > c_\alpha | C''_{pk} = C) = \alpha$. The decision making rule to be

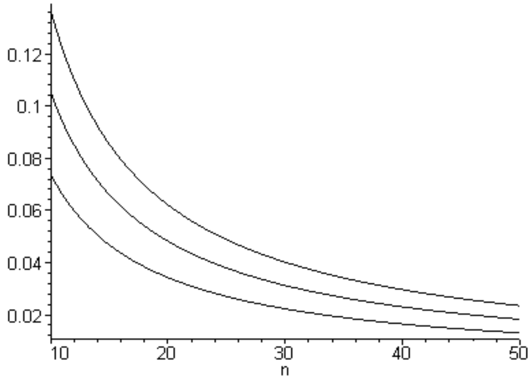

Fig. 7. Bias plot of $\hat{C}''_{pk}$ (vs. n) for $\xi = 0.5$, $d_\ell = 5/6$, $d_u = 5/4$ with $b = 3$, 4, and 5 (from bottom to top in the plot).

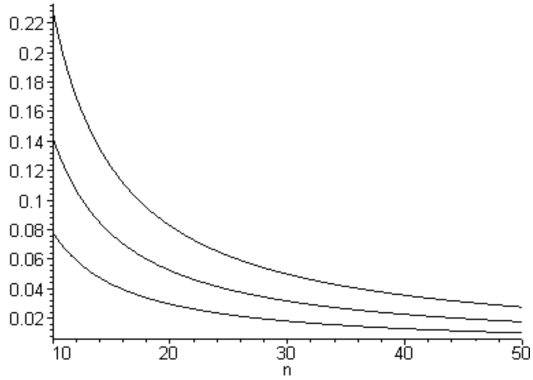

Fig. 8. MSE plot of $\hat{C}''_{pk}$ (vs. n) for $\xi = 0.5$, $d_\ell = 5/6$, $d_u = 5/4$ with $b = 3$, 4, and 5 (from bottom to top in the plot).

used is then that, for given values of risk α and sample size n, the process will be considered capable if $\hat{C}''_{pk} > c_\alpha$ and incapable if $\hat{C}''_{pk} \le c_\alpha$.

We note that by setting $\xi = (\mu - T)/\sigma$ and $b = d^*/\sigma$, the index C''_{pk} can be rewritten as $C''_{pk} = [b + \xi/\max\{1, r\}]/3$ for $\xi < 0$ and $C''_{pk} = [b - \xi \min\{1, r\}]/3$ for $\xi \ge 0$ where $r = D_\ell/D_u$. Hence, the value of C''_{pk} can be calculated given values of ξ, b, and r. For example, if $(\xi, b, r) = (-1, 3, 3/2)$ then $C''_{pk} = [3 + (-1)/\max\{1, 3/2\}]/3 = 7/9 = 0.7778$. If $C''_{pk} = C$, we have $b = 3C - \xi/\max\{1, r\}$ for $\xi < 0$ and $b = 3C + \xi \min\{1, r\}$ for $\xi \ge 0$. In addition, since $B = n^{1/2}(d^*/\sigma)$ and $b = d^*/\sigma$ then $B^2 = nb^2$. Therefore, if $C''_{pk} = C$ then

$$B^2 = \begin{cases} n(3C - \xi/\max\{1, r\})^2, & \xi < 0 \\ n(3C + \xi \min\{1, r\})^2, & \xi \ge 0. \end{cases} \tag{11}$$

Table 4a. Critical values c_α for C = 1.00 with $|\xi|$ = 0.0(0.1)1.0, and sample sizes n = 10(10)100 for T = m and α-risk = 0.01

$\lvert\xi\rvert$	n = 10	20	30	40	50	60	70	80	90	100
0.00	1.926	1.500	1.369	1.303	1.262	1.233	1.212	1.195	1.182	1.171
0.10	1.988	1.545	1.409	1.340	1.297	1.266	1.244	1.226	1.212	1.200
0.20	2.037	1.578	1.436	1.363	1.316	1.284	1.260	1.240	1.225	1.212
0.30	2.075	1.599	1.451	1.374	1.325	1.290	1.265	1.244	1.228	1.214
0.40	2.101	1.612	1.458	1.378	1.327	1.292	1.265	1.245	1.228	1.214
0.50	2.119	1.618	1.460	1.379	1.328	1.292	1.266	1.245	1.228	1.214
0.60	2.130	1.620	1.461	1.379	1.328	1.292	1.266	1.245	1.228	1.214
0.70	2.136	1.621	1.461	1.379	1.328	1.292	1.266	1.245	1.228	1.214
0.80	2.139	1.621	1.461	1.379	1.328	1.292	1.266	1.245	1.228	1.214
0.90	2.141	1.621	1.461	1.379	1.328	1.292	1.266	1.245	1.228	1.214
1.00	2.141	1.621	1.461	1.379	1.328	1.292	1.266	1.245	1.228	1.214

Table 4b. Critical values c_α for C = 1.00 with $|\xi|$ = 0.0(0.1)1.0, and sample sizes n = 10(10)100 for T = m and α-risk = 0.05

$\lvert\xi\rvert$	n = 10	20	30	40	50	60	70	80	90	100
0.00	1.523	1.300	1.227	1.188	1.163	1.146	1.133	1.123	1.115	1.108
0.10	1.572	1.339	1.262	1.221	1.195	1.177	1.163	1.152	1.143	1.135
0.20	1.610	1.367	1.284	1.240	1.212	1.191	1.176	1.164	1.153	1.145
0.30	1.639	1.384	1.296	1.249	1.218	1.196	1.180	1.166	1.155	1.146
0.40	1.659	1.393	1.301	1.252	1.220	1.197	1.180	1.167	1.156	1.147
0.50	1.671	1.397	1.303	1.252	1.220	1.197	1.180	1.167	1.156	1.147
0.60	1.679	1.399	1.303	1.252	1.220	1.197	1.180	1.167	1.156	1.147
0.70	1.683	1.399	1.303	1.252	1.220	1.197	1.180	1.167	1.156	1.147
0.80	1.685	1.399	1.303	1.252	1.220	1.197	1.180	1.167	1.156	1.147
0.90	1.686	1.399	1.303	1.252	1.220	1.197	1.180	1.167	1.156	1.147
1.00	1.686	1.399	1.303	1.252	1.220	1.197	1.180	1.167	1.156	1.147

We can use the central chi-square distribution and the normal distribution to find the critical value c_α satisfying $P(\hat{C}''_{pk} > c_\alpha | C''_{pk} = C) = \alpha$, i.e., $1 - F_{\hat{C}''_{pk}}(c_\alpha) = \alpha$ given $C''_{pk} = C$. We note that c_α is larger than zero in general, hence we can find c_α by Eq. (9)

$$\int_0^{B^2} F_K(L(c_\alpha, y)) f_Y(y) dy = \alpha, \tag{12}$$

where B^2 is given in Eq. (11) and $L(c_\alpha, y) = (n-1)(B - y^{1/2})^2/(9nc_\alpha^2)$.

We point out that if $T = m$ (symmetric tolerance) then C''_{pk} reduces to C_{pk} and $\hat{C}''_{pk}$ reduces to $\hat{C}_{pk}$. We note that the critical values c_α for $\xi = \xi_0$ and $\xi = -\xi_0$ are the same when $T = m$ (for the proof see Appendix B). Tables 4a–7b display the critical values c_α for $C = 1.00, 1.33, 1.66$, and 2.00 with sample sizes $n = 10(10)100$, $|\xi| = 0.0(0.1)1.0$ and α-risk $= 0.01$ and 0.05 for $T = m$.

To test if the process meets the capability (quality) requirement, we first determine the value of C and the α-risk. Since both the process parameters μ and σ are unknown, then parameter $\xi = (\mu - T)/\sigma$ is also unknown. But, we

Table 5a. Critical values c_α for $C = 1.33$ with $|\xi| = 0.0(0.1)1.0$, and sample sizes $n = 10(10)100$ for $T = m$ and α-risk $= 0.01$

$\|\xi\|$	n = 10	20	30	40	50	60	70	80	90	100
0.00	2.606	2.017	1.837	1.746	1.689	1.650	1.620	1.597	1.579	1.564
0.10	2.667	2.062	1.877	1.782	1.723	1.682	1.652	1.628	1.608	1.592
0.20	2.716	2.094	1.902	1.804	1.742	1.698	1.666	1.641	1.620	1.603
0.30	2.751	2.113	1.915	1.813	1.749	1.704	1.670	1.644	1.622	1.605
0.40	2.776	2.124	1.921	1.817	1.751	1.705	1.671	1.644	1.623	1.605
0.50	2.792	2.129	1.923	1.817	1.751	1.705	1.671	1.644	1.623	1.605
0.60	2.802	2.131	1.924	1.817	1.751	1.705	1.671	1.644	1.623	1.605
0.70	2.807	2.132	1.924	1.817	1.751	1.705	1.671	1.644	1.623	1.605
0.80	2.809	2.132	1.924	1.817	1.751	1.705	1.671	1.644	1.623	1.605
0.90	2.811	2.132	1.924	1.817	1.751	1.705	1.671	1.644	1.623	1.605
1.00	2.811	2.132	1.924	1.817	1.751	1.705	1.671	1.644	1.623	1.605

Table 5b. Critical values c_α for $C = 1.33$ with $|\xi| = 0.0(0.1)1.0$, and sample sizes $n = 10(10)100$ for $T = m$ and α-risk $= 0.05$

$\|\xi\|$	n = 10	20	30	40	50	60	70	80	90	100
0.00	2.062	1.750	1.647	1.593	1.558	1.534	1.516	1.502	1.491	1.481
0.10	2.111	1.789	1.682	1.626	1.590	1.564	1.545	1.530	1.518	1.508
0.20	2.149	1.815	1.703	1.644	1.605	1.578	1.557	1.541	1.528	1.516
0.30	2.176	1.831	1.714	1.651	1.611	1.582	1.560	1.543	1.529	1.518
0.40	2.194	1.840	1.719	1.654	1.612	1.583	1.561	1.544	1.529	1.518
0.50	2.206	1.843	1.720	1.654	1.612	1.583	1.561	1.544	1.529	1.518
0.60	2.213	1.845	1.720	1.654	1.612	1.583	1.561	1.544	1.529	1.518
0.70	2.217	1.845	1.720	1.654	1.612	1.583	1.561	1.544	1.529	1.518
0.80	2.219	1.845	1.720	1.654	1.612	1.583	1.561	1.544	1.529	1.518
0.90	2.219	1.845	1.720	1.654	1.612	1.583	1.561	1.544	1.529	1.518
1.00	2.220	1.845	1.720	1.654	1.612	1.583	1.561	1.544	1.529	1.518

Table 6a. Critical values c_α for $C = 1.66$ with $|\xi| = 0.0(0.1)1.0$, and sample sizes $n = 10(10)100$ for $T = m$ and α-risk $= 0.01$

$\|\xi\|$	n = 10	20	30	40	50	60	70	80	90	100
0.00	3.287	2.535	2.306	2.190	2.117	2.067	2.029	2.000	1.977	1.958
0.10	3.349	2.580	2.346	2.226	2.151	2.099	2.060	2.030	2.006	1.986
0.20	3.396	2.611	2.370	2.246	2.168	2.114	2.074	2.042	2.016	1.995
0.30	3.431	2.630	2.382	2.255	2.175	2.119	2.077	2.044	2.018	1.997
0.40	3.455	2.639	2.387	2.258	2.177	2.120	2.078	2.045	2.019	1.997
0.50	3.469	2.644	2.389	2.258	2.177	2.120	2.078	2.045	2.019	1.997
0.60	3.479	2.646	2.389	2.258	2.177	2.120	2.078	2.045	2.019	1.997
0.70	3.483	2.646	2.389	2.258	2.177	2.120	2.078	2.045	2.019	1.997
0.80	3.486	2.646	2.389	2.258	2.177	2.120	2.078	2.045	2.019	1.997
0.90	3.487	2.646	2.389	2.258	2.177	2.120	2.078	2.045	2.019	1.997
1.00	3.487	2.646	2.389	2.258	2.177	2.120	2.078	2.045	2.019	1.997

can estimate ξ by calculating the value $\hat{\xi} = (\overline{X} - T)/S$ from the sample. If the estimated value $\hat{C}''_{pk}$ is larger than the critical value $c_\alpha(\hat{C}''_{pk} > c_\alpha)$, then we conclude that the process meets the capability requirement $(C''_{pk} > C)$. Otherwise, we do not have sufficient information to conclude that the process

Table 6b. Critical values c_α for $C = 1.66$ with $|\xi| = 0.0(0.1)1.0$, and sample sizes $n = 10(10)100$ for $T = m$ and α-risk $= 0.05$

$\|\xi\|$	n = 10	20	30	40	50	60	70	80	90	100
0.00	2.603	2.200	2.068	1.998	1.954	1.923	1.900	1.882	1.867	1.855
0.10	2.651	2.239	2.103	2.031	1.985	1.953	1.928	1.909	1.894	1.881
0.20	2.689	2.265	2.124	2.048	2.000	1.966	1.940	1.919	1.903	1.889
0.30	2.715	2.280	2.134	2.056	2.005	1.970	1.943	1.922	1.904	1.889
0.40	2.733	2.288	2.138	2.058	2.006	1.970	1.943	1.922	1.904	1.890
0.50	2.744	2.291	2.139	2.058	2.006	1.970	1.943	1.922	1.904	1.890
0.60	2.751	2.293	2.140	2.058	2.006	1.970	1.943	1.922	1.904	1.890
0.70	2.754	2.293	2.140	2.058	2.006	1.970	1.943	1.922	1.904	1.890
0.80	2.756	2.293	2.140	2.058	2.006	1.970	1.943	1.922	1.904	1.890
0.90	2.756	2.293	2.140	2.058	2.006	1.970	1.943	1.922	1.904	1.890
1.00	2.757	2.294	2.140	2.058	2.006	1.970	1.943	1.922	1.904	1.890

Table 7a. Critical values c_α for $C = 2.00$ with $|\xi| = 0.0(0.1)1.0$, and sample sizes $n = 10(10)100$ for $T = m$ and α-risk $= 0.01$.

$\|\xi\|$	n = 10	20	30	40	50	60	70	80	90	100
0.00	3.991	3.070	2.790	2.647	2.559	2.498	2.451	2.416	2.387	2.364
0.10	4.052	3.115	2.829	2.683	2.592	2.529	2.482	2.446	2.416	2.391
0.20	4.099	3.145	2.853	2.703	2.609	2.543	2.495	2.456	2.426	2.400
0.30	4.133	3.163	2.864	2.711	2.615	2.548	2.498	2.459	2.427	2.401
0.40	4.156	3.172	2.869	2.713	2.616	2.549	2.498	2.459	2.428	2.402
0.50	4.170	3.176	2.870	2.714	2.616	2.549	2.498	2.459	2.428	2.402
0.60	4.178	3.178	2.871	2.714	2.616	2.549	2.498	2.459	2.428	2.402
0.70	4.183	3.179	2.871	2.714	2.616	2.549	2.498	2.459	2.428	2.402
0.80	4.185	3.179	2.871	2.714	2.616	2.549	2.498	2.459	2.428	2.402
0.90	4.186	3.179	2.871	2.714	2.616	2.549	2.498	2.459	2.428	2.402
1.00	4.186	3.179	2.871	2.714	2.616	2.549	2.498	2.459	2.428	2.402

Table 7b. Critical values c_α for $C = 2.00$ with $|\xi| = 0.0(0.1)1.0$, and sample sizes $n = 10(10)100$ for $T = m$ and α-risk $= 0.05$.

$\|\xi\|$	n = 10	20	30	40	50	60	70	80	90	100
0.00	3.160	2.665	2.502	2.417	2.362	2.324	2.295	2.273	2.255	2.240
0.10	3.209	2.704	2.537	2.449	2.393	2.353	2.324	2.301	2.282	2.266
0.20	3.246	2.729	2.557	2.466	2.407	2.366	2.334	2.310	2.290	2.273
0.30	3.272	2.744	2.567	2.472	2.412	2.369	2.337	2.312	2.291	2.274
0.40	3.289	2.751	2.571	2.474	2.413	2.370	2.337	2.312	2.291	2.274
0.50	3.300	2.755	2.572	2.475	2.413	2.370	2.337	2.312	2.291	2.274
0.60	3.306	2.756	2.572	2.475	2.413	2.370	2.337	2.312	2.291	2.274
0.70	3.309	2.756	2.572	2.475	2.413	2.370	2.337	2.312	2.291	2.274
0.80	3.311	2.756	2.572	2.475	2.413	2.370	2.337	2.312	2.291	2.274
0.90	3.312	2.756	2.572	2.475	2.413	2.370	2.337	2.312	2.291	2.274
1.00	3.312	2.756	2.572	2.475	2.413	2.370	2.337	2.312	2.291	2.274

meets the present capability requirement. In this case, we would believe that $C''_{pk} \le C$ (the process is incapable).

We also can calculate the *p*-value, i.e. the probability that $\hat{C}''_{pk}$ exceed the observed estimated index given the values of C, $\xi = (\mu - T)/\sigma$, $r = D_\ell/D_u$, and sample size n, and then compare this probability with the significance level α. If the estimated index value is c_0, given the values of C, ξ, r, and sample size n, then the *p*-value can be calculated as:

$$p\text{-value} = P(\hat{C}''_{pk} > c_0 | C''_{pk} = C) = 1 - F_{\hat{C}''_{pk}}(c_0)$$
$$= \int_0^{B^2} F_K(L(c_0, y)) \frac{1}{2\sqrt{y}} \left(\frac{D_u}{d^*} \phi[\frac{D_u}{d^*}\sqrt{y} - \xi\sqrt{n}] + \frac{D_\ell}{d^*} \phi[\frac{D_\ell}{d^*}\sqrt{y} + \xi\sqrt{n}] \right) dy, \tag{13}$$

where $D_u/d^* = 1/\min\{1, r\}$, $D_\ell/d^* = \max\{1, r\}$, B^2 is given in Eq. (11) and $L(c_0, y) = (n-1)(B - y^{1/2})^2/(9nc_0^2)$. The numerical calculations can be easily carried out using the *Maple* computer software, to integrate the function based on the central chi-square distribution and the normal distribution. If the *p*-value is smaller than the α-risk, than we conclude that the process meets the capability requirement ($C''_{pk} > C$). Otherwise, we do not have sufficient information to conclude that the process meets the present capability requirement. In this case, we would believe that $C''_{pk} \leq C$ (the process is incapable).

As an example, we consider the following normally distributed process with asymmetric specification tolerances $LSL = 20$, $T = 26.5$, and $USL = 32$. We note that $d = (USL - LSL)/2 = 6$, $D_\ell = T - LSL = 6.5$, $D_u = USL - T = 5.5$, $d^* = \min\{D_\ell, D_u\} = 5.5$, $r = D_\ell/D_u = 1.18$. To test if the process meets the capability (quality) requirement, we first determine $C = 1.33$, i.e., we define a process with $C''_{pk} > 1.33$ is capable. If the sample size $n = 100$, the sample mean $\overline{X} = 27$, and the sample standard deviation $S = 1.10$. We can calculate $\hat{A}^* = \max\{d^*(\overline{X} - T)/D_u, d^*(T - \overline{X})/D_\ell\} = 0.5$, $\hat{\xi} = (\overline{X} - T)/S = 0.45$, and $\hat{C}''_{pk} = 1.515$. We find the corresponding *p*-value is 0.055 using the *Maple* computer software to calculate Eq. (13). We conclude that the process meets the capability requirement if the α-risk is set larger than 0.055. If the α-risk is set smaller than 0.055, we do not have sufficient information to conclude that the process meets the present capability requirement.

5 An application example

The example presented in the following concerns with the capability of a process, which produces electronic telecommunication amplifiers (see Pearn *et al.* (2001)). The original data and a complete description of this process are given in Juran Institute (1990). The quality characteristic of interest is the gain (the boosting ability) of an amplifier. The design of the amplifiers had called for a gain of 10 decibels (dB) and allowed the amplifiers to be considered acceptable if the gain fell between 7.75 dB and 12.25 dB, i.e. (LSL, T,USL) = (7.75, 10, 12.25). A sample of the gains of 120 amplifiers was taken by the quality improvement team to estimate the capability of the manufacturing process producing the amplifiers. Chou *et al.* (1998) noted that the data follow a non-Normal distribution. The data were then fitted by an S_B distribution. They also transformed the data to approximate Normality using the estimated transformation

Table 8. The original amplifier gain data

8.1	10.4	8.8	9.7	7.8	9.9	11.7	8.0	9.3	9.0
8.2	8.9	10.1	9.4	9.2	7.9	9.5	10.9	7.8	8.3
9.1	8.4	9.6	11.1	7.9	8.5	8.7	7.8	10.5	8.5
11.5	8.0	7.9	8.3	8.7	10.0	9.4	9.0	9.2	10.7
9.3	9.7	8.7	8.2	8.9	8.6	9.5	9.4	8.8	8.3
8.4	9.1	10.1	7.8	8.1	8.8	8.0	9.2	8.4	7.8
7.9	8.5	9.2	8.7	10.2	7.9	9.8	8.3	9.0	9.6
9.9	10.6	8.6	9.4	8.8	8.2	10.5	9.7	9.1	8.0
8.7	9.8	8.5	8.9	9.1	8.4	8.1	9.5	8.7	9.3
8.1	10.1	9.6	8.3	8.0	9.8	9.0	8.9	8.1	9.7
8.5	8.2	9.0	10.2	9.5	8.3	8.9	9.1	10.3	8.4
8.6	9.2	8.5	9.6	9.0	10.7	8.6	10.0	8.8	8.6

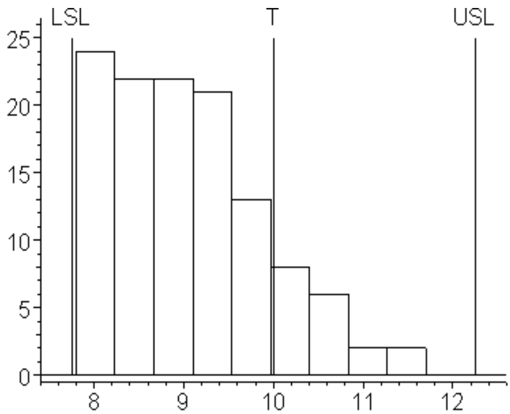

Fig. 9. Histogram of the 120 untransformed amplifier gain data with (LSL, T, USL) = (7.75, 10, 12.25).

$$Z = 0.96 + 0.98\ ln\left(\frac{X - 7.59}{4.68 + 7.59 - X}\right). \tag{14}$$

We note that a significant error may be introduced if someone use the original specification limits, (LSL, T, USL) $= (7.75, 10, 12.25)$, to evaluate the quality through the transformed data. Using the estimated transformation Eq. (14), we have the transformed specification $(\mathrm{LSL}', \mathrm{T}', \mathrm{USL}') = (-2.31, 1.00, 5.06)$ as well as the transformed data.

Table 8 displays the sample of the original gains of 120 amplifiers listed in Juran Institute (1990). A histogram of the data, with the specification limits, is given in Figure 9. Table 9 displays the corresponding transformed amplifier gain data, using the estimated transformation in Eq. (14). A histogram of the transformed data, with the transformed specification limits, is given in Figure 10. We may now apply a normal-based SPC procedure to the transformed data. We note that the transformed specification (LSL′, T′, USL′) is asymmetric. Therefore, we apply the proposed generalization C''_{pk} to the transformed data. To test if the quality of the amplifiers meets the quality

Table 9. The transformed amplifier gain data

−1.1	1.4	−0.1	0.8	−2.0	0.9	2.9	−1.3	0.4	0.1
−0.9	0.0	1.1	0.5	0.3	−1.6	0.6	1.8	−2.0	−0.7
0.2	−0.6	0.7	2.0	−1.6	−0.4	−0.2	−2.0	1.4	−0.4
2.6	−1.3	−1.6	−0.7	−0.2	1.0	0.5	0.1	0.3	1.6
0.4	0.8	−0.2	−0.9	0.0	−0.3	0.6	0.5	−0.1	−0.7
−0.6	0.2	1.1	−2.0	−1.1	−0.1	−1.3	0.3	−0.6	−2.0
−1.6	−0.4	0.3	−0.2	1.2	−1.6	0.9	−0.7	0.1	0.7
0.9	1.5	−0.3	0.5	−0.1	−0.9	1.4	0.8	0.2	−1.3
−0.2	0.9	−0.4	0.0	0.2	−0.6	−1.1	0.6	−0.2	0.4
−1.1	1.1	0.7	−0.7	−1.3	0.9	0.1	0.0	−1.1	0.8
−0.4	−0.9	0.1	1.2	0.6	−0.7	0.0	0.2	1.3	−0.6
−0.3	0.3	−0.4	0.7	0.1	1.6	−0.3	1.0	−0.1	−0.3

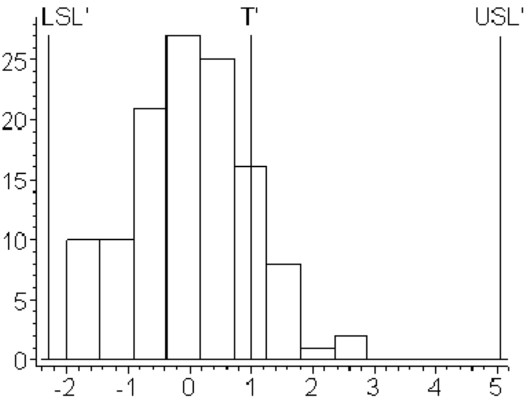

Fig. 10. Histogram of the 120 transformed amplifier gain data with (LSL', T', USL') = (-2.31, 1.00, 5.06).

requirement, we first determine $C = 1.00$, i.e., we define a process with $C''_{pk} > 1.00$ is capable. We then calculate $d = (USL' - LSL')/2 = 3.685$, $d^* = \min\{D_u, D_\ell\} = \min\{4.06, 3.31\} = 3.31$, $n = 120$, $\overline{Z} = \sum_{i=1}^{n} Z_i/n = 0.000713$, $S^2 = \sum_{i=1}^{n}(Z_i - \overline{Z})^2/(n-1) = 0.985$, $S = 0.993$, $\hat{A}^* = \max\{d^*(\overline{Z} - T')/D_u, d^*(T' - \overline{Z})/D_\ell\} = \max\{-0.815, 0.999\} = 0.999$, $\hat{\xi} = (\overline{Z} - T')/S = -1.007$, and $\hat{C}''_{pk} = (d^* - \hat{A}^*)/(3S) = 0.776$. We then find the corresponding *p*-value be 0.9999 using the *Maple* computer software to calculate Eq. (13). Obviously, the quality of the amplifiers does not meet the quality requirement: $C''_{pk} > 1.00$.

While all the 120 amplifiers fell within the specification limits, the low value of $\hat{C}''_{pk}$ shows that the average quality of the amplifiers significantly deviates from the target value, which is unsatisfactory causing the communication failed. The quality improvement team could now concentrate their investigations to find problems causing the manufacturing line incapable, and find ways to make the process average closer to the target value. Some quality improvement activities involving Taguchi's parameter designs should be initiated to identify the significant factors causing the process failing to cluster around the target value.

6 Conclusions

Pearn and Chen (1998) proposed the new generalization C''_{pk} which was shown to be superior to other existing generalizations of C_{pk} for processes with asymmetric tolerances. In this paper, we investigated the relation between the fraction nonconforming and the value of C''_{pk}. We also obtained the cumulative distribution function and the probability density function of the estimated index $\hat{C}''_{pk}$ for processes with normal distributions. We showed that the cumulative distribution function and the probability density function of $\hat{C}''_{pk}$ can be expressed in terms of a mixture of the chi-square distribution and the normal distribution. Consequently, the complexity for analyzing the statistical properties of $\hat{C}''_{pk}$ is greatly simplified. We also analyzed the bias and the MSE of the estimated index $\hat{C}''_{pk}$ for normally distributed processes. Furthermore, we also developed a decision making rule, based on the natural estimator $\hat{C}''_{pk}$. The function of *p*-value was given and the numerical calculations of *p*-value can be easily carried out using mathematical computer softwares, e.g., *Mathematica*, *Maple*, and *MatLab*. Therefore, the practitioners can use the proposed decision making rules to test whether the process with asymmetric tolerance is capable or not.

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Appendix A: Derivation of Eq. (9)

Under the assumption of normality, the cumulative distribution function and the probability density function of $\hat{C}''_{pk}$ can be derived as follows.

[Case I]: For $x > 0$, using the technique of conditioning $\hat{C}''_{pk}$ on Y in Eq. (6), we may obtain

$$\begin{aligned}
F_{\hat{C}''_{pk}}(x) &= P(\hat{C}''_{pk} \le x) = P\left(\frac{\sqrt{n-1}(B-\sqrt{Y})}{3\sqrt{nK}} \le x\right) \\
&= 1 - P\left(\sqrt{nK} < \frac{\sqrt{n-1}(B-\sqrt{Y})}{3x}\right) \\
&= 1 - \int_0^\infty P\left(\sqrt{nK} < \frac{\sqrt{n-1}(B-\sqrt{y})}{3x}\right) f_Y(y)\,dy.
\end{aligned}$$

Since K is distributed as χ^2_{n-1}, then $P\left(\sqrt{nK} < \frac{\sqrt{n-1}(B-\sqrt{y})}{3x}\right) = 0$ for $y > B^2$ and $x > 0$. Hence,

$$\begin{aligned}
F_{\hat{C}''_{pk}}(x) &= 1 - \int_0^{B^2} P\left(\sqrt{nK} < \frac{\sqrt{n-1}(B-\sqrt{y})}{3x}\right) f_Y(y)dy \\
&= 1 - \int_0^{B^2} P(K < L(x,y))\, f_Y(y)dy \\
&= 1 - \int_0^{B^2} F_K(L(x,y))\, f_Y(y)dy, \quad \text{for } x > 0,
\end{aligned} \tag{A1}$$

where $B = n^{1/2}(d^*/\sigma)$, $L(x,y) = (n-1)(B - y^{1/2})^2/(9nx^2)$, $F_K(\cdot)$ is the cumulative distribution function of K, and $f_Y(\cdot)$ is the probability density function of Y expressed as Eq. (8).

[Case II]: Since K is distributed as χ^2_{n-1}, then $P\left(\sqrt{nK} \le \frac{\sqrt{n-1}(B-\sqrt{y})}{3x}\right) = 0$ for $x < 0$ and $y < B^2$. Hence,

$$\begin{aligned}
F_{\hat{C}''_{pk}}(x) &= \int_{B^2}^\infty P\left(\sqrt{nK} \le \frac{\sqrt{n-1}(B-\sqrt{y})}{3x}\right) f_Y(y)\,dy \\
&= \int_{B^2}^\infty P(K \le L(x,y))\, f_Y(y)\,dy \\
&= \int_{B^2}^\infty F_K(L(x,y))\, f_Y(y)\,dy, \quad \text{for } x < 0.
\end{aligned} \tag{A2}$$

[Case III]: For $x = 0$, we have

$$\begin{aligned} F_{\hat{C}''_{pk}}(0) &= P(\hat{C}''_{pk} \le 0) = P\left(\frac{\sqrt{n-1}(B-\sqrt{Y})}{3\sqrt{nK}} \le 0\right) \\ &= P\left(B - \sqrt{Y} \le 0\right) \\ &= 1 - P\left(Y < B^2\right) \\ &= 1 - F_Y(B^2), \end{aligned} \tag{A3}$$

where $F_Y(\cdot)$ is the cumulative distribution function of Y expressed as Eq. (7).

Combining Eqs. (A1), (A2) and (A3), we obtain Eq. (9) for the cumulative distribution function of $\hat{C}''_{pk}$. Taking the derivative of the cumulative distribution function of $\hat{C}''_{pk}$ with Leibniz's rule and changing the variable with $t = y/B^2$, we obtain the probability density function of $\hat{C}''_{pk}$ in Eq. (10).

Appendix B: Symmetry property of ξ

Following we will show that given the same values of C, n, and α the critical values c_α for $\xi = \xi_0$ and $\xi = -\xi_0$ are the same when $T = m$.

Since if $T = m$ then $r = D_\ell/D_u = 1$ and by Eq. (11) $B^2 = D^2 = n(d/\sigma)^2 = n(3C + |\xi|)^2$ given $C''_{pk} = C$. Furthermore, $\delta = n^{1/2}(\mu - T)/\sigma = n^{1/2}\xi$ and $f_Y(y)$ expressed as Eq. (8) reduces to

$$\begin{aligned} f_Y(y) &= \frac{1}{2\sqrt{y}}\left(\phi[\sqrt{y} - \sqrt{n}\,\xi] + \phi[\sqrt{y} + \sqrt{n}\,\xi]\right) \\ &= \frac{1}{2\sqrt{y}}\left(\phi[\sqrt{y} - \sqrt{n}\,|\xi|] + \phi[\sqrt{y} + \sqrt{n}\,|\xi|]\right), \text{ for } T = m. \end{aligned} \tag{A4}$$

Therefore, the Eq. (12) reduces to

$$\int_0^{D^2} F_K(L(c_\alpha, y)) f_Y(y)\, dy = \alpha, \tag{A5}$$

where $L(c_\alpha, y) = (n-1)(D - y^{1/2})^2/(9nc_\alpha^2)$ with $D = n^{1/2}(3C + |\xi|)$ given $C''_{pk} = C$ and $f_Y(y)$ expressed as Eq. (A4). We get the same equation if we substitute ξ by ξ_0 and $(-\xi_0)$ into Eq. (A5) given the same values of C, n, and α. Therefore, Eq. (A5) is an even function of ξ for case $T = m$. Hence, given the same values of C, n, and α the critical values c_α for $\xi = \xi_0$ and $\xi = -\xi_0$ are the same when $T = m$.